Iterative Convolution-Thresholding method for interface related optimization problems

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Introduction & Background

Interface related optimization problems arise from many applications including image segmentation; heat sink design; structure, shape and topology optimization; optimal composite material; fluid network design; and so on.

- The Mathematical Problem

$$\min_{\Omega_0} \mathcal{J}(\Omega_0, \Theta)$$
$$s.t.; C(\Omega_0, \Theta) = 0.$$

where

- $J(\Omega_0, \Theta)$: objective functional
- $\Omega_0 \subset \Omega$: domain to be optimized in a computational domain Ω
- Θ: possible state variables (e.g. velocity field, temperature, etc.)

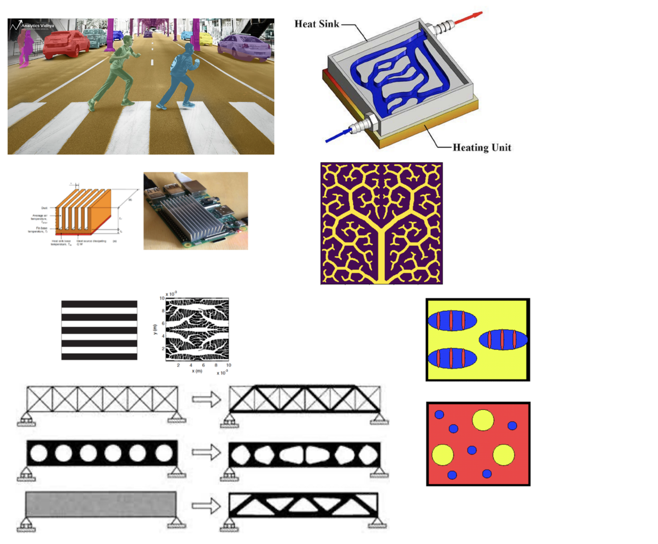


Figure 1: Interface related optimization problems

Mathematical modelling and simulation of the problem:

Representation of the interface
Approximation of the problem
Numerical methods

We propose a framework algorithm based on the indicator function representation.

The mathematical problem

- The general formula:

$$(u^*, \Theta^*) = \arg\min_{u\in\mathcal{B}, \Theta\in\mathcal{S}} \mathcal{E} := \sum_{i=1}^n \int_{\Omega_i} F_i(\Theta_1, \dots, \Theta_n) dx + \lambda \sum_{i=1}^n |\partial\Omega_i|.$$

- $\mathcal{B} = \left\{ (u_1, u_2, \dots, u_n) \in BV(\Omega, \mathbb{R}^n) \mid u_i = \{0, 1\} \text{ and } \sum_{i=1}^n u_i(x) = 1 \right\}$,
- $\mathcal{S}$: admissible set of $\Theta = (\Theta_1, \Theta_2, \dots, \Theta_n)$.

- The indicator function representation:

$$u_i(x) = \chi_{\Omega_i}(x) := \begin{cases} 1 \text{ if } x \in \Omega_i, \\ 0 \text{ otherwise,} \end{cases} \quad i \in [n].$$

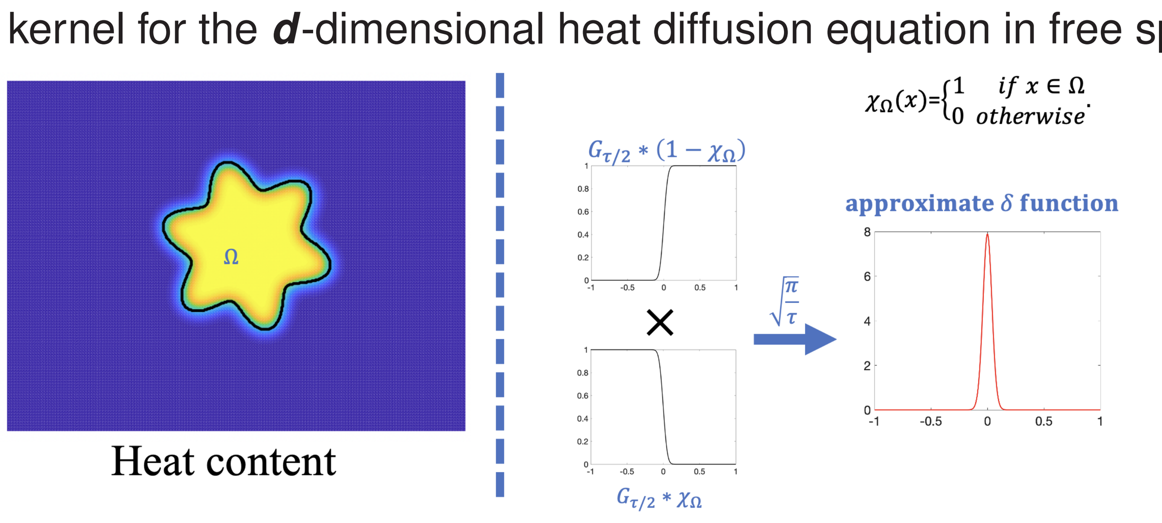
- The approximation: When $\tau \ll 1$, the length of $\partial\Omega_i \cap \partial\Omega_j$ can be approximated by [Esedoglu + Otto, CPAM, 2015]

$$|\partial\Omega_i \cap \partial\Omega_j| \approx \sqrt{\frac{\pi}{\tau}} \int_\Omega u_i G_\tau * u_j \, dx,$$

where $*$ represents convolution and

$$G_{\delta t}(x) = \frac{1}{(4\pi\tau)^{d/2}} exp(-\frac{|x|^2}{4\tau})$$

is the heat kernel for the d-dimensional heat diffusion equation in free space.



- The approximate problem:

$$\mathcal{E}^\tau(u, \Theta) = \sum_{i=1}^n \left[\int_\Omega u_i F_i(\Theta_1, \dots, \Theta_n) \, dx + \lambda \sum_{j=1, j\neq i}^n \sqrt{\frac{\pi}{\tau}} \int_\Omega u_i G_\tau * u_j \, dx \right].$$

Derivation of the method

Apply the coordinate descent method to minimize $\mathcal{E}^\tau(u, \Theta)$; that is, starting from an initial guess: u^0, we find the minimizers iteratively in the following order:

$$\Theta^0, u^0, \Theta^1, \dots, u^k, \Theta^k, \dots$$

where

$$\Theta^k = \min_{\Theta\in\mathcal{S}} \sum_{i=1}^n \int_\Omega u_i^k F_i(\Theta_1, \dots, \Theta_n) dx \quad (1)$$

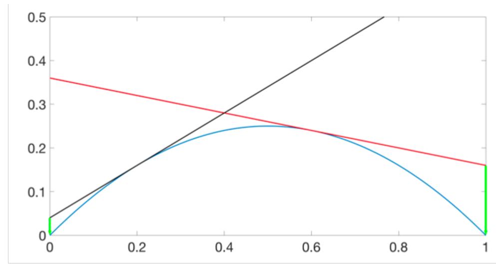
$$u^{k+1} = \min_{u\in\mathcal{B}} \sum_{i=1}^n \int_\Omega u_i F_i(\Theta_1^k, \dots, \Theta_n^k) dx + \lambda \sum_{j=1, j\neq i}^n \sqrt{\frac{\pi}{\tau}} \int_\Omega u_i G_\tau * u_j \, dx \quad (2)$$

Since $\mathcal{E}^\tau(u, \Theta)$ is concave in u, solution to (2) can be approximated by solving the relaxed and linearized problem.

$$u^{k+1} = \arg\min_{u\in\mathcal{K}} \mathcal{L}^\tau(f, \Theta^k, u^k, u)$$

Derivation of the method

- $\mathcal{L}^\tau(f, \Theta^k, u^k, u) = \sum_{i=1}^n \int_\Omega u_i \phi_i^k \, dx$
- $\phi_i^k = F_i(\Theta_1^k, \dots, \Theta_n^k) + \lambda \sum_{j=1, j\neq i}^n \sqrt{\frac{\pi}{\tau}} G_\tau * u_j^k$



- $\mathcal{K} = \left\{ (u_1, u_2, \dots, u_n) \in BV(\Omega, \mathbb{R}^n) \mid u_i \in [0, 1], \text{ and } \sum_{i=1}^n u_i(x) = 1 \right\}$

The Iterative Convolution-Thresholding Method (ICTM)

1. For the fixed u^s, find

$$\Theta^s = \arg\min_{\Theta\in\mathcal{S}} \sum_{i=1}^n \int_\Omega u_i^s F_i(\Theta_1, \dots, \Theta_n) dx.$$

2. For $i \in [n]$, evaluate

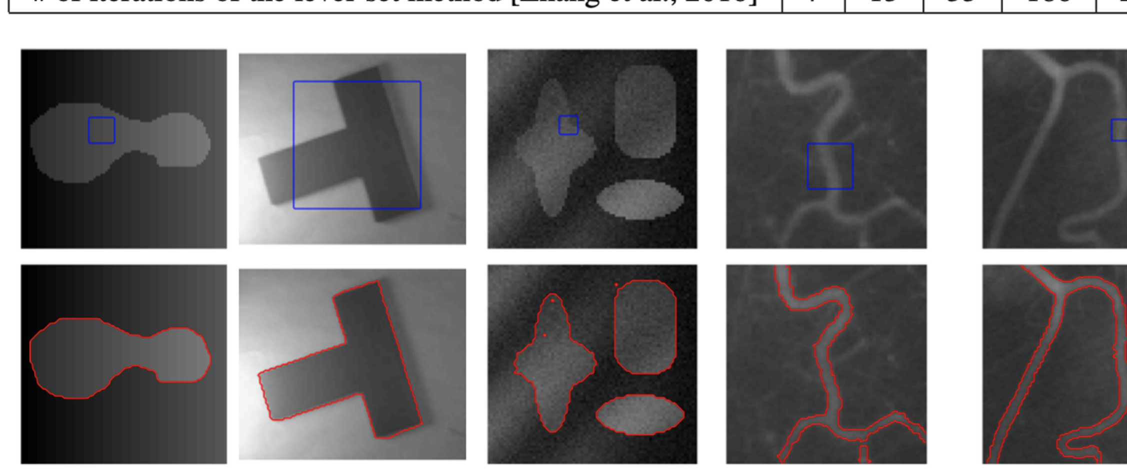
$$\phi_i^s = F_i(\Theta_1^s, \dots, \Theta_n^s) + \lambda \sum_{j=1, j\neq i}^n \sqrt{\frac{\pi}{\tau}} G_\tau * u_j^s.$$

3. For $i \in [n]$, set

$$u_i^{s+1}(x) = \begin{cases} 1 & \text{if } i = \min\{\arg\min_{\ell\in[n]} \phi_\ell^s\}, \\ 0 & \text{otherwise.} \end{cases}$$

Numerical Results

# of iterations of the ICTM	8	7	7	7	7
# of iterations of the level-set method [Zhang et al., 2016]	7	13	35	186	239



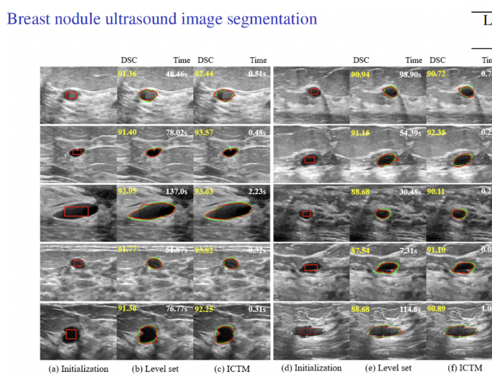
# of iterations of the ICTM	13	25	43	28	47
# of iterations of the level-set method [Li et al., 2008]	29	256	131	117	209

- Region based $\mathcal{E} = \sum_{i=1}^n \int_{\Omega_i} F_i(f, \Theta_1, \dots, \Theta_n) \, dx + \sum_{i=1}^n \lambda_i |\partial\Omega_i|$
- Edge based $\min_\phi \int_\Omega g\delta(\phi)|\nabla\phi| \, d\mathbf{x}, \quad g := \frac{1}{1 + |\nabla G_\sigma * I|^2}$

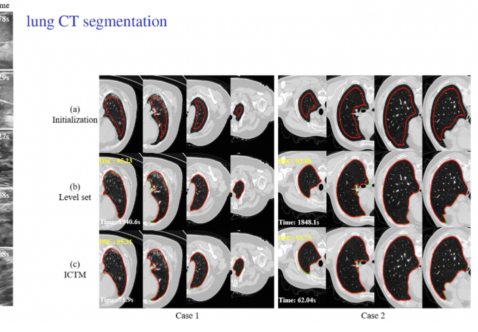
Liver lesion segmentation in MR

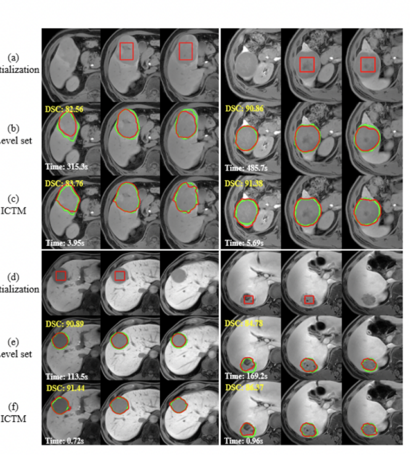
Dataset	Method	DSC (%) ↑	# of Iteration ↓	Acceleration
Breast Ultrasound	Level set	87.75 ± 5.34	719 ± 398	12.4 × faster
	ICTM (ours)	**89.10 ± 3.99**	**58 ± 48**	
COVID-19 CT	Level set	92.18 ± 2.35	7826 ± 2200	3.86 × faster
	ICTM (ours)	**92.35 ± 3.21**	**1384 ± 586**	
Liver Tumor MR	Level set	85.79 ± 6.32	475 ± 378	5.65 × faster
	ICTM (ours)	**86.44 ± 6.61**	**123 ± 145**	

Breast nodule ultrasound image segmentation

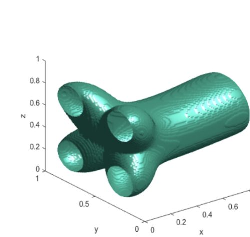


lung CT segmentation





$$\min_{(\Omega_0,\mathbf{u})} J_0(\Omega_0,\mathbf{u}) = \int_\Omega \left(\frac{\mu}{2}|D\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{f}\right) d\mathbf{x} + \gamma|\Gamma|$$

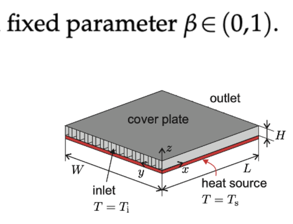


$$\begin{aligned} \nabla\cdot\mathbf{u} &= 0, \text{ in } \Omega, \\ \nabla p - \nabla\cdot(\mu\nabla\mathbf{u}) &= \mathbf{f}, \text{ in } \Omega_0, \\ \mathbf{u} &= 0, \text{ in } \Omega\backslash\Omega_0, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{u}_D, \text{ on } \partial\Omega, \\ |\Omega_0| &= \beta|\Omega| \text{ with a fixed parameter } \beta\in(0,1). \end{aligned}$$

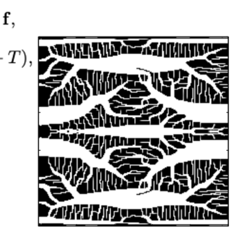
$$\min_{\mathbf{u},\Omega_f,\Omega_s} \frac{1}{2}\int_\Omega \alpha\mathbf{u}\cdot\mathbf{u} + 2\mu\epsilon(\mathbf{u}) : \epsilon(\mathbf{u}) \, dx - \int_\Omega g\cdot\mathbf{u} \, dx - \int_{\Gamma_\mathbf{t}} \mathbf{t}\cdot\mathbf{u} \, dS$$

$$\begin{aligned} &\alpha\mathbf{u} - 2\nabla\cdot(\mu\epsilon(\mathbf{u})) + \nabla p = g, \\ &\nabla\cdot\mathbf{u} = s, \\ &\mathbf{u}|_{\Gamma_D} = \mathbf{u}_D, \\ &-p\mathbf{n} + 2\mu\epsilon(\mathbf{u})\mathbf{n}|_{\Gamma_\mathbf{t}} = \mathbf{t} \end{aligned}$$

$$\min_{\Omega_0} - \int_\Omega \kappa\frac{K_{de}}{H^2}(T_s - T) \, d\mathbf{x}$$



$$\begin{aligned} &\nabla\cdot\mathbf{u} = 0, \\ &K_{cm}\rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \nabla\cdot(\mu\nabla\mathbf{u}) + \alpha\mathbf{u} = \mathbf{f}, \\ &K_{ce}\rho c\nabla\cdot(\mathbf{u}T) - \nabla\cdot(\kappa\nabla T) = \kappa\frac{K_{de}}{H^2}(T_s - T), \\ &\mathbf{u} = \mathbf{u}_D, \\ &\mu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\mathbf{n} = \mathbf{g}, \\ &T = t_D, \\ &\frac{\partial T}{\partial\mathbf{n}} = 0, \end{aligned}$$



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