Linearly and Nonlinearly Preconditioning Techniques for Elasticity with Applications in Arterial Walls

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Introduction (Anisotropic Hyperelastic Model)

Arterial wall can be modeled by a nearly incompressible, anisotropic and hyperelastic equation that allows large deformation.

- Energy Functional

$$\psi = \psi^{iso}(C) + \psi^{vol}(C) + \psi^{ti}(C, M^{(i)}), \quad (1)$$

where C is Cauthy-Green tensor, $M^{(i)}$ are the structural tensors.

- Momentum Equation

$$\mathbf{div} P = -f, \quad (2)$$

where $P = FS$, $S = \frac{\partial \psi}{\partial C}$.

- Principal Invariants

$$I_1 := \mathrm{tr}\ C, \quad I_2 := \mathrm{tr}\ [\mathrm{cof} C], \quad I_3 := \det\ C,$$

$$J_4^{(i)} := \mathrm{tr}[CM^{(i)}], \quad J_5^{(i)} := \mathrm{tr}[C^2 M^{(i)}].$$

with $\psi^{iso} = \psi^{iso}(I_i)$, $\psi^{vol} = \psi^{vol}(I_3)$ and $\psi^{ti} = \psi^{ti}(I_i, J_j^{(i)})$.

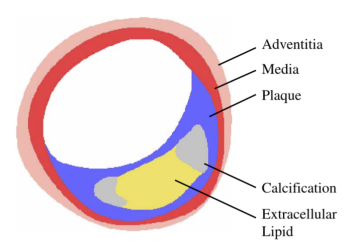


Figure 1: Cross-section of artery

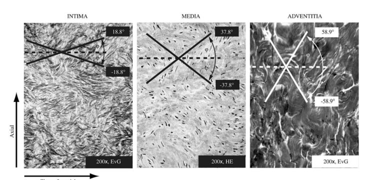


Figure 2: Collagen fibre reinforcement

The performance of Inexact Newton methods (IN) and the linear solvers degrade in the cases of

Large Deformation; Near Incompressibility; High Anisotropy.

We propose some robust linearly iterative solvers and a nonlinearly preconditioned Newton's method.

Optimal Multilevel Linear Preconditioners

Denote the nonlinear system discretized from (2) by

$$F(u^*) = 0$$

where $F : \mathbb{R}^n \mapsto \mathbb{R}^n$.

- A serial of Jacobian systems need to solve approximately

$$\|F(u^{(k)}) + F'(u^{(k)})p^{(k)}\| \leq \eta_k \|F(u^{(k)})\|, \quad (3)$$

where $u^{(k)}$ is a approximate solution, and $\eta_k \in [0, 1)$ is a scalar that determines how accurately the Jacobian system needs to be solved.

- The multilevel preconditioners based on domain decomposition are given in [1] as

$$B = I_H^h A_H^{-1} (I_H^h)^T + \sum_{i=1}^{J} \iota_i A_i^{-1} \iota_i^T. \quad (4)$$

where A_H is the nonconforming discretization ($\mathcal{P}_2$-$\mathcal{P}_0$) for linear elastic operator; i.e.

$$\langle A_H w_H, v_H \rangle := 2\tilde{\mu}(\epsilon(w_H), \epsilon(v_H)) + \tilde{\lambda}(P_0^H \mathrm{div} w_H, P_0^H \mathrm{div} v_H)\ \forall w_H, v_H \in W_H, \quad (5)$$

- The uniform convergence is proved for the case of linear elasticity:

Theorem (Condition number estimate)

The condition number of BF' satisfies

$$\mathrm{cond}(BF') \leq c(1 + N_c)\frac{H^2}{\delta^2},$$

where c is independent to mesh size and material parameters.

IN Method with Nonlinear Elimination Preconditioner

- High nonlinearity ≈ High residual

$$F(u_0 + \delta u) = F(u_0) + F'(u_0)(\delta u) + \frac{1}{2}F''(u_0)(\delta u, \delta u) + \mathcal{O}(\|\delta u\|^3).$$

- Find "bad" dofs set S_b from $S = \{1, \cdots, n\}$, according to the residual

$$V_b = \{v \mid v = (v_1, \cdots, v_n)^T \in \mathbb{R}^n, v_k = 0, \text{ if } k \notin S_b\}.$$

- Given an approximation u, NE finds correction by solving $u_b \in V_b$ such that

$$F_b(u_b) := R_b F(u_b + u) = 0.$$

Algorithm (IN-NE)

Step 1. Compute the next approximate solution $u^{(k+1)}$ by solving

$$F(u) = 0$$

with one step of IN iteration using $u^{(k)}$ as the initial guess.

Step 2. (Nonlinearity checking)

2.1 If $\|F(u^{(k+1)})\| < \varrho_1 \|F(u^{(k)})\|$, go to Step 1.

2.2 Finding "bad" d.o.f. by

$$S_b := \{j \in S \mid |F_j(u^{(k+1)})| > \varrho_2 \|F(u^{(k+1)})\|_\infty\}.$$

And extend S_b to S_b^δ by adding the neighbor d.o.f..

2.3 If $\#(S_b^\delta) < \varrho_3 n$, go to Step 3. Otherwise, go to Step 1.

Step 3. Compute the correction $u_b^\delta \in V_b$ by solving the subproblem approximately

$$F_b^\delta(u_b^\delta) := R_b^\delta F(u_b^\delta + u^{(k+1)}) = 0,$$

with an initial guess $u_b^\delta = 0$.Update $u^{(k+1)} \leftarrow u_b^\delta + u^{(k+1)}$. Go to Step 1.

Test Examples

We consider the polyconvex energy functional

$$\psi_A = \psi^{isochoric} + \psi^{volumetric} + \psi^{ti}$$

$$:= c_1\left(\frac{I_1}{I_3^{1/3}} - 3\right) + \epsilon_1\left(I_3^{\epsilon_2} + \frac{1}{I_3^{\epsilon_2}} - 2\right) + \sum_{i=1}^{2}\alpha_1\left\langle I_1 J_4^{(i)} - J_5^{(i)} - 2\right\rangle^{\alpha_2}. \quad (6)$$

Based on the parameter sets of the model ψ_A in Table. 1, we propose three test examples to investigate the performance of our algorithms for the case of large deformation, near incompressibility and high anisotropy.

Set	Layer	c_1	ϵ_1	ϵ_2(-)	α_1	α_2	Purpose
L	–	1.e3	1.e3	1.0	0.0	0.0	Deformations by different pulls
C1	–	1.e3	1.e3	1.0	0.0	0.0	Different penalties for compressiblity
C2	–	1.e3	1.e4	1.0	0.0	0.0	
C3	–	1.e3	1.e5	1.0	0.0	0.0	
A1	Adv.	7.5	100.0	20.0	1.5e10	20.0	Anisotropic arterial walls
	Med.	17.5	100.0	50.0	5.0e5	7.0	
A2	Adv.	6.6	23.9	10	1503.0	6.3	
	Med.	17.5	499.8	2.4	30001.9	5.1	
A3	Adv.	7.8	70.0	8.5	1503.0	6.3	
	Med.	9.2	360.0	9.0	30001.9	5.1	

Table 1: Model parameter sets of ψ_A.

Numerical Results

- The simulation results for the three examples are depicted as follows

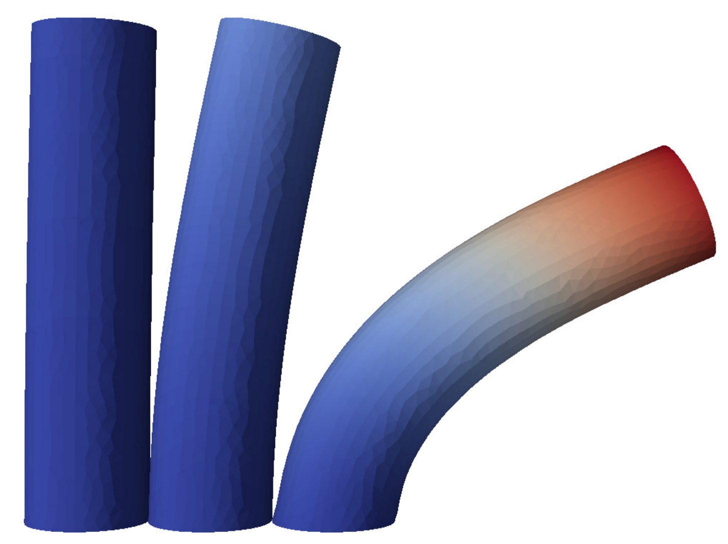


Figure 3: Deformations by different pulls.

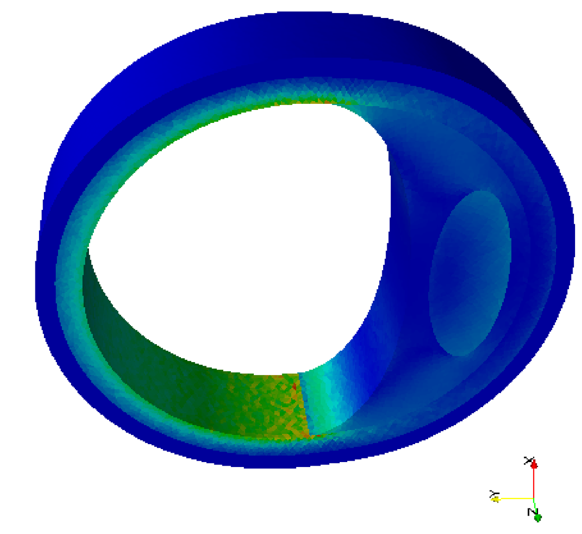
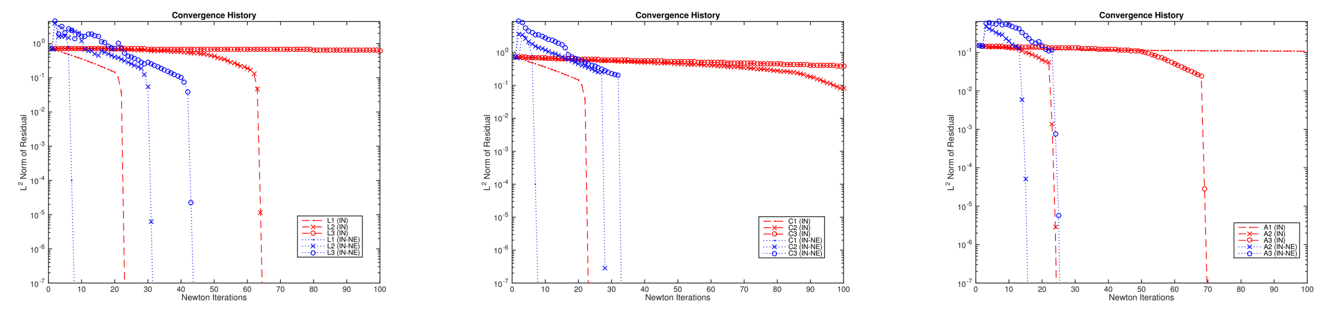


Figure 4: von Mises stress

- The convergence history



(a) Example L (b) Example C (c) Example A

Figure 5: Convergence history of IN and IN-NE

- The parameter-sensitivity tests

$\varrho_2 = .9, \varrho_3 = .3$			
ϱ_1	.9	.95	.98
Global Newton iterations	23	23	24
Total Newton iterations of NE	40	25	23
$\rho_1 = .95, \rho_3 = .3$			
ϱ_2	.8	.9	.95
Global Newton iterations	24	23	25
Total Newton iterations of NE	31	25	23
$\rho_1 = .95, \rho_2 = .9$			
ϱ_3	.1	.2	.3
Global Newton iterations	54	23	23
Total Newton iterations of NE	15	25	25

Table 2: Number of iterations of IN-NE with respect to different pre-chosen parameters.

Set	Poisson's Ratio
C1	0.125
C2	0.452
C3	0.495

Table 3: Poisson's ratio of materials C1, C2 and C3.

mesh \ δ	2	3	4
m_0	15	23	23
m_1	58	36	26

Table 4: Mesh refinement, Set A2, $\rho_0 = .9$, $\rho_1 = .9$, $\rho_2 = .25$.

Reference & Acknowledgements

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